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OPTIMUM CONTROL WITH DESIRED INPUT PIECE-WISE CONTINUOUS

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ABSTRACT

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A closed form solution of the optimum control law is derived utilizing the Pontryagin maximum principle. The desired value of the input is piece-wise continuous. The derivation presented for one discontinuity can be extended to any number of piece-wise discontinuities. The principle of dynamic programming holds, that is, the system is optimum for the remaining time no matter what disturbance has occurred in the past. The method is applicable to linear constant coefficient systems of any order.

The application of the general method is shown by an example in which the optimum control law for a nuclear reactor start-up is determined. Compared to the method of state variables the closed form solution given by the optimum control law has the advantage of being less susceptible to random noise which may be present in the instrumentation. A closed form solution of the control law for a second order system is derived for any reference trajectory with continuous second derivatives.

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INTRODUCTION

Previous investigators(1)*,(2) have considered application of the Pontryagin maximum principle to various specialized cases of second order systems for common error criterion. The applications were limited to establishing predetermined trajectories and switching surfaces, and did not produce a control system which could reduce the effects due to unpredicted external disturbances such as noise. The objective of this paper is to demonstrate that an optimum feedback controller can be obtained for time invariant linear systems by the Pontryagin maximum principle.

The optimum control system will be defined as the control system which minimizes the integral squared error from the present time t to a final time T_2. The principle of dynamic programming applies, that is, no matter what the control has been prior to the present time, the control will be optimal for all future time up to the final time T_2. The desired input $M(\sigma)$ is piece-wise continuous as shown in Figure 1. The subscripts a and b refer to desired inputs and outputs for two different time intervals. Thus,

$$e(t) = \int_t^{T_1} \left\{ \xi[Q_a(\sigma) - q(\sigma)]^2 + \nu[M_a(\sigma) - m(\sigma)]^2 \right\} d\sigma$$

$$+ \int_{T_1}^{T_2} \left\{ \xi[Q_b(\sigma) - q(\sigma)]^2 + \nu[M_b(\sigma) - m(\sigma)]^2 \right\} d\sigma \quad (1)$$

where: $e(t)$ = error criterion,

*Superscript numbers in parenthesis refer to references at the end of the paper.

$q(\sigma)$ = output signal,

$Q(\sigma)$ = desired signal,

$m(\sigma)$ = input signal,

$M(\sigma)$ = desired input, which is piece-wise continuous, see Figure 1,

σ = dummy time variable representing future time,

t = present time,

T_2 = termination time of the control system,

ξ ,ν = weighting factors.

Linear constant coefficient systems are described by the following equation,

$$a_n \frac{d^n q(\sigma)}{d\sigma^n} = - \sum_{i=o}^{n-1} a_i \frac{d^i q(\sigma)}{d\sigma^i} + m(\sigma) \tag{2}$$

where $a_i (i=0,1,2, \ldots n)$ are constants and by definition,

$$\frac{d^0 q(\sigma)}{d\sigma^0} = q(\sigma)$$

There are n boundary conditions for the values of q and its derivatives at the present time $\sigma = t$. The final value of the output is not fixed, but the natural boundary conditions will be imposed.

OPTIMUM CONTROL LAW

An auxiliary time variable μ will be defined such that μ is measured from the present time t as shown in Figure 1, thus,

$$\sigma = t + \mu \tag{3}$$

The time t will be treated as a parameter. Equation (2) becomes,

$$a_n \frac{d^n q(t+\mu)}{d\mu^n} = - \sum_{i=0}^{n-1} a_i \frac{d^i q(t+\mu)}{d\mu^i} + m(t+\mu) \tag{4}$$

and the error criterion becomes,

$$e(t) = \int_{\mu=0}^{\mu=T_1-t} \left\{ \xi[Q_a(t+\mu)-q(t+\mu)]^2 + \nu[M_a(t+\mu)-m(t+\mu)]^2 \right\} d\mu$$
$$+ \int_{\mu=T_1-t}^{\mu=T_2-t} \left\{ \xi[Q_b(t+\mu)-q(t+\mu)]^2 + \nu[M_b(t+\mu)-m(t+\mu)]^2 \right\} d\mu \tag{5}$$

Converting to the Pontryagin form we define,

$$u(t;\mu) = m(t+\mu)$$
$$x_1(t;\mu) = q(t+\mu)$$
$$x_{i+1}(t;\mu) = \frac{dx_i(t;\mu)}{d\mu} \qquad i = 1,2, \ldots (n-1)$$

$$x_{n+1}(t;\mu) = \int_{\tau=0}^{T_1-t} \left\{ \xi[Q_a(t+\tau)-q(t+\tau)]^2 + \nu[M_a(t+\tau)-m(t+\tau)]^2 \right\} d\tau$$
$$+ \int_{\tau=T_1-t}^{\mu} \left\{ \xi[Q_b(t+\tau)-q(t+\tau)]^2 + \nu[M_b(t+\tau)-m(t+\tau)]^2 \right\} d\tau \tag{6}$$

By successive differentiation,

$$\frac{d^i x_1(t;\mu)}{d\mu^i} = x_{i+1}(t;\mu) \qquad i=1,2, \ldots n-1$$
$$\frac{d^n x_1(t;\mu)}{d\mu^n} = \frac{dx_n(t;\mu)}{d\mu} \tag{7}$$

With Equations (6) and (7), Equations (4) and (5) become, the system of first order equations,

$$\frac{dx_i(t;\mu)}{d\mu} = x_{i+1}(t;\mu) = f_i \qquad i = 1,2, \ldots (n-1)$$

$$\frac{dx_n(t;\mu)}{d\mu} = \frac{1}{a_n}\left[- \sum_{i=o}^{n-1} a_i x_{i+1}(t;\mu) + u(t;\mu)\right] = f_n$$

$$\frac{dx_{n+1}(t;\mu)}{d\mu} = \xi[Q(t+\mu) - x_1(t;\mu)]^2 + \nu[M(t+\mu) - u(t;\mu)]^2 = f_{n+1} \tag{8}$$

where: $Q = Q_a$ & $M = M_a$ if $t < T_1$,

$Q = Q_b$ & $M = M_b$ if $T_1 < t < T_2$.

The optimum system is defined as the system for which,

$$S(t) = \sum_{i=1}^{n+1} c_i x_i(t;\mu = T_2 - t) \tag{9}$$

where: $c_i = 0$ for $i \neq n+1$ and $c_{n+1} = 1$,
is a minimum with respect to $u(t;\mu)$. The Hamiltonian, given in Appendix A, is,

$$H = \sum_{i=1}^{n+1} p_i(t;\mu)\ f_i \tag{10}$$

where $p_i(t;\mu)$ is the auxiliary variable. From Equations (8),

$$H = \sum_{i=1}^{n-1} p_i(t;\mu)x_{i+1}(t;\mu) + \frac{p_n(t;\mu)}{a_n}\left[- \sum_{i=o}^{n-1} a_i x_{i+1}(t;\mu)+u(t;\mu)\right]$$
$$+ p_{n+1}(t;\mu)\left\{\xi[Q(t+\mu)-x_1(t;\mu)]^2 + \nu\ [M(t+\mu) - u(t;\mu)]^2\right\} \tag{11}$$

For $S(t)$ to be a minimum, H must be a maximum. For the unsaturated case,

$$\frac{\partial H}{\partial u^*} = 0 = \frac{p_n(t;\mu)}{a_n} - 2\nu p_{n+1}(t;\mu)[M(t+\mu) - u^*(t;\mu)] \tag{12}$$

or

$$u^*(t;\mu) = M(t+\mu) - \frac{1}{2\nu a_n} \frac{p_n(t;\mu)}{p_{n+1}(t;\mu)} \tag{13}$$

where the asterisk denotes the optimum condition.

DIFFERENTIAL EQUATIONS FOR THE AUXILIARY VARIABLES

The previous section developed the optimum control law in terms of auxiliary variables. In order to complete the derivation of the optimum control law, it is necessary to develop the differential equations for the auxiliary variables. From Equation (A-6) of APPENDIX A,

$$\frac{dp_1(t;\mu)}{d\mu} = \frac{a_o}{a_n} p_n(t;\mu) + 2\xi p_{n+1}(t;\mu)\,[Q(t+\mu) - x_1(t;\mu)]$$

$$\frac{dp_i(t;\mu)}{d\mu} = \frac{a_{i-1}}{a_n} p_n(t;\mu) - p_{i-1}(t;\mu) \qquad i = 2,3 \ldots n \tag{14}$$

$$\frac{dp_{n+1}(t;\mu)}{d\mu} = 0$$

From the last Equation of (14), $p_{n+1}(t;\mu)$ is not a function of μ, but may be an arbitrary function of the parameter t, $p_{n+1}(t;\mu) = f(t)$, such that the end point condition, Equation (A-4) is satisfied for all t. For Equation (A-4) to be satisfied for all t, $p_{n+1}(t;\mu)$ must be constant. With $c_{n+1} = 1$ from Equation (9) we have,

$$p_{n+1}(t;\mu) = -1 \tag{15}$$

Differentiating the i^{th} Equation of (14) i-1 times gives,

$$\frac{d^i p_i(t;\mu)}{d\mu^i} = \frac{a_{i-1}}{a_n} \frac{d^{i-1} p_n(t;\mu)}{d\mu^{i-1}} - \frac{d^{i-1} p_{i-1}(t;\mu)}{d\mu^{i-1}} \tag{16}$$

Differentiating the $(i-1)^{th}$ equation i-2 times, and substituting into (16),

$$\frac{d^i p_i(t;\mu)}{d\mu^i} = \frac{a_{i-1}}{a_n} \frac{d^{i-1} p_n(t;\mu)}{d\mu^{i-1}} - \frac{a_{i-2}}{a_n} \frac{d^{i-2} p_n(t;\mu)}{d\mu^{i-2}} + \frac{d^{i-2} p_{i-2}(t;\mu)}{d\mu^{i-2}} \tag{17}$$

The recursion formula leads to,

$$\frac{d^n p_n(t;\mu)}{d\mu^n} = \sum_{i=1}^{n-1} (-1)^{i+1} \frac{a_{n-i}}{a_n} \frac{d^{n-i} p_n(t;\mu)}{d\mu^{n-i}} + (-1)^{n+1} \left\{ \frac{a_o}{a_n} p_n(t;\mu) - 2\xi [Q(t+\mu) - x_1(t;\mu)] \right\} \tag{18}$$

or

$$x_1(t;\mu) = Q(t+\mu) + \frac{(-1)^n}{2\xi} \sum_{i=0}^{n} (-1)^{i+1} \frac{a_{n-i}}{a_n} \frac{d^{n-i} p_n(t;\mu)}{d\mu^{n-i}} \tag{19}$$

From the definition of $x_1(t;\mu)$ and $u(t;\mu)$, Equation (4) is,

$$\sum_{j=0}^{n} a_j \frac{d^j x_1(t;\mu)}{d\mu^j} = u(t;\mu) \tag{20}$$

Substituting Equation (19) into (20) and using (13) and (15),

$$\sum_{j=0}^{n} a_j \left\{ \frac{d^j Q(t+\mu)}{d\mu^j} + \frac{(-1)^n}{2\xi} \sum_{i=0}^{n} (-1)^{i+1} \frac{a_{n-i}}{a_n} \frac{d^{n+j-i} p_n(t;\mu)}{d\mu^{n+j-i}} \right\} = M(t+\mu) + \frac{1}{2\nu a_n} p_n(t;\mu) \tag{21}$$

The desired input and output should in general satisfy the original system differential equation. This insures that minimization of the error criterion is compatible with the system dynamics. If the original system differential equation is not satisfied by the desired input and output, it is not possible to reduce the error criterion to zero. With this requirement,

$$\sum_{j=0}^{n} a_j \frac{d^j Q(t+\mu)}{d\mu^j} = M(t+\mu) \tag{22}$$

and Equation (21) becomes,

$$\sum_{j=0}^{n} \sum_{i=0}^{n} (-1)^{i+1} a_j a_{n-i} \frac{d^{n+j-i} p_n(t;\mu)}{d\mu^{n+j-i}} = (-1)^n \omega^2 p_n(t;\mu) \tag{23}$$

where $\omega^2 = \xi/\nu$.

All the odd derivatives of $p_n(t;\mu)$ are zero in Equation (23). For example, the equation for a first order system is,

$$\frac{d^2 p_1(t;\mu)}{d\mu^2} - \left[\frac{a_o^2 + \omega^2}{a_1^2}\right] p_1(t;\mu) = 0 \tag{24}$$

and for a second order system,

$$\frac{d^4 p_2(t;\mu)}{d\mu^4} + \left[2 \frac{a_o}{a_2} - \left(\frac{a_1}{a_2}\right)^2\right] \frac{d^2 p_2(t;\mu)}{d\mu^2} + \left[\frac{a_o^2 + \omega^2}{a_2^2}\right] p_2(t;\mu) = 0 \tag{25}$$

Equation (23) is of order 2n. There are n boundary conditions at each present time for the values of x_1. The remaining n boundary conditions are from the transversality conditions for the case with natural boundary conditions, which are expressed in terms of the adjoint variables by Equation (A-4).

SOLUTIONS WITH DESIRED INPUT PIECE-WISE CONTINUOUS

As an example, the start-up and regulation of a nuclear reactor is given by Equation (C-5) in APPENDIX C as a special case of Equation (2) with $n = 1$. For the nuclear reactor, $a_o = 0$, $a_1 = 1$ and the desired input and output are given by,

$$\left.\begin{aligned} Q_a(t) &= at \\ M_a(t) &= a \end{aligned}\right\} \quad 0 \leq t < T_1 \qquad \left.\begin{aligned} Q_b(t) &= aT_1 \\ M_b(t) &= 0 \end{aligned}\right\} \quad T_1 \leq t \leq T_2 \tag{26}$$

There is a discontinuity of the desired input M at $t = T_1$, therefore $M(t)$ is piece-wise continuous. The physical significance of this desired program is discussed in APPENDIX C.

For the time region $T_1 \leq t \leq T_2$ the solution to Equation (24) is,

$$p_1(t;\mu) = A_b \cosh(\omega\mu) + B_b \sinh(\omega\mu) \tag{27}$$

where A_b and B_b are arbitrary constants to be determined. From Equations (A-4) and (9) one natural boundary condition, which applies to the free terminal point problem, is,

$$p_1(t;\mu = T_2 - t) = - c_1 = 0 \tag{28}$$

which requires,

$$A_b = - B_b \tanh[\omega(T_2-t)] \tag{29}$$

or,

$$p_1(t;\mu) = B_b \left\{ \sinh(\omega\mu) - \tanh[\omega(T_2-t)] \cosh(\omega\mu) \right\} \tag{30}$$

substituting Equation (30) into (19) with $n = 1$,

$$x_1(t;\mu) = Q_b(t+\mu) + \frac{B_b}{2} \frac{\omega}{\xi} \frac{\cosh[\omega(T_2-(\mu+t))]}{\cosh[\omega(T_2 -t)]} \tag{31}$$

Using the equivalence between $x_1(t;\mu)$ and $q(t+\mu)$ expressed by Equation (6), Equation (31) with (3) becomes,

$$q(\sigma) = Q_b(\sigma) + \frac{B_b}{2}\,\frac{\omega}{\xi}\,\frac{\cosh[\omega(T_2-\sigma)]}{\cosh[\omega(T_2-t)]} \tag{32}$$

The constant B_b is evaluated by requiring $q(\sigma)$ to equal the measured value $q^*(t)$ at $\sigma = t$,

$$B_b = -\,2\,\frac{\xi}{\omega}\,[Q_b(t) - q^*(t)] \tag{33}$$

Substituting Equations (30) and (33) into (13) with the change in variables given by (3) and (6),

$$\begin{aligned} m^*(\sigma) = M_b(\sigma) - \omega[Q_b(t) - q^*(t)]\,\Big\{&\sinh[\omega(\sigma-t)] \\ &- \tanh[\omega(T_2-t)]\,\cosh[\omega(\sigma-t)]\Big\} \end{aligned} \tag{34}$$

and in particular at the present time $\sigma = t$,

$$m^*(t) = M_b(t) + \omega[Q_b(t) - q^*(t)]\,\tanh[\omega(T_2-t)] \tag{35}$$

$$T_1 \le t \le T_2$$

The equivalent expression for $m^*(t)$ in the first time interval $t \le T_1$ is derived in APPENDIX B.

With the desired input and output given by Equations (26), the control laws for the two time intervals become,

$$m^*(t) = \omega[aT_1 - q^*(t)]\,\tanh[\omega(T_2-t)] \qquad T_1 \le t \le T_2 \tag{36}$$

$$\begin{aligned} m^*(t) = a\,\Big\{&1 - \cosh[\omega(T_2-T_1)]\,\mathrm{sech}[\omega(T_2-t)]\Big\} \\ &+ \omega[at - q^*(t)]\,\tanh[\omega(T_2-t)] \qquad t < T_1 \end{aligned} \tag{37}$$

Equations (36) and (37) are identical to the results previously obtained by the authors[3], using the methods of calculus of variations and dynamic programming.

RANDOM NOISE CONSIDERATIONS

The optimum control law given by Equations (36) and (37) can be implemented as shown by the block diagram in Figure 2. Random noise,[4] ${}^{k}n(t)$, may be introduced into the feedback signal, $k_q(t)$, by the instrumentation such as the neutron detectors. In this section, the amount of error introduced into the optimum control system shown in Figure 2 will be compared to the amount of error introduced into a typical state variable switching system shown in Figure 3.

The actual system output, ${}^{k}y(t)$, is related to the measured system output as follows,

$$ {}^{k}q(t) = {}^{k}y(t) + {}^{k}n(t) \tag{38} $$

The average random noise is assumed to be zero, $\widetilde{{}^{k}n(t)} \rightarrow 0$. The low pass filter characteristic of the process causes all actual system output records to approach a single value, $y(t)$, i.e.,

$$ {}^{k}y(t) \rightarrow y(t). \tag{39} $$

The actual output approaches the average measured system output,

$$ y(t) \rightarrow \widetilde{{}^{k}q(t)} = x_1(t) \tag{40} $$

where the variable $x_1(t)$ may be considered to be deterministic for the optimization problem. The variable ${}^{k}q(t)$ is reserved for the random process.

The formulation of the optimum control law changes very little for this case. In Equations (4) and (5) the quantity $q(t+\mu)$ becomes ${}^{k}q(t+\mu)$ where,

$$ {}^{k}q(t+\mu) = {}^{k}y(t+\mu) + {}^{k}n(t+\mu) = x_1(t;\mu) \tag{41} $$

which is identical to the second line of Equation (6) if $q(t+\mu)$ becomes ${}^{k}q(t+\mu)$. It can be proved that the previous analysis holds also for this particular (though common) type of random process.

The error will be defined as the difference between the desired system output and the actual system output,

$$ {}^{k}\varepsilon(t) = \begin{cases} at - {}^{k}y(t) & \text{for } t < T_1 \\ aT_1 - {}^{k}y(t) & \text{for } T_1 \leq t \leq T_2 \end{cases} \tag{42} $$

The autocorrelation function of the random noise is not zero, thus, $[{}^{k}y(t)]^2 \neq [{}^{k}q(t)]^2$. From Figure 2 and Equations (36) and (37),

$$ \frac{d}{dt}[{}^{k}y(t)] - a = b(t)[at - {}^{k}y(t) - {}^{k}n(t)] - a\,\frac{\operatorname{sech}[\omega(T_2-t)]}{\operatorname{sech}[\omega(T_2-T_1)]} \tag{43a} $$

for $t < T_1$, and

$$ \frac{d}{dt}[{}^{k}y(t)] = b(t)[aT_1 - {}^{k}y(t) - {}^{k}n(t)] \tag{43b} $$

for $T_1 \leq t \leq T_2$, where $b(t) = \omega\tanh[\omega(T_2-t)]$. Because only the effect of the noise is under consideration, the last term in Equation (43a) may be dropped, which with Equations (42) leads to the following result,

$$\frac{d}{dt}\,[{}^{k}\varepsilon(t)] + b(t)[{}^{k}\varepsilon(t)] = b(t)[{}^{k}n(t)] \tag{44}$$

The solution of Equation (44) in integral form is,(5)

$${}^{k}\varepsilon(t) = {}^{k}\varepsilon(0)\eta(t) + \eta(t)\int_{0}^{t} \frac{b(\tau)\,{}^{k}n(\tau)}{\eta(\tau)}\,d\tau \tag{45}$$

where,

$$\eta(t) = \exp\left[-\int_{0}^{t} b(\tau)d\tau\right]$$

If the output signal and noise are not correlated, the mean-squared value of $\varepsilon(T_2)$ is given by,(5)

$$\overline{[{}^{k}\varepsilon(T_2)]^2} = \overline{[{}^{k}\varepsilon(0)]^2}\,\eta^2(T_2) + \eta^2(T_2)\int_{0}^{T_2} \frac{b(\alpha)}{\eta(\alpha)}\,d\alpha \int_{0}^{T_2} \frac{b(\beta)}{\eta(\beta)}\,\gamma_{nn}(\alpha,\beta)\,d\beta \tag{46}$$

where $\gamma_{nn}(\alpha,\beta)$ is the autocorrelation function of the noise. When the random process is stationary, the autocorrelation function becomes $\gamma_{nn}(\alpha-\beta)$.

For most of the operation of the optimum control system, $\omega(T_2-t)$ will be large, therefore $\tanh[\omega(T_2-t)]$ will be approximately unity except near the neighborhood of $t = T_2$. Thus the transfer function from Equation (44) becomes,

$$\frac{{}^{k}\varepsilon(s)}{{}^{k}n(s)} = \frac{\omega}{s+\omega}$$

For a stationary process, a typical autocorrelation function for the noise may be given as,

$$\gamma_{nn}(\tau) = Ke^{-\zeta|\tau|} \tag{47}$$

where K and ζ are constants. The autocorrelation function of the output, $\gamma_{\varepsilon\varepsilon}(\tau)$, is found from the Weiner-Khinchin[4] and the input-output relations to be,

$$\gamma_{\varepsilon\varepsilon}(\tau) = \frac{K\zeta\omega^2}{\omega^2-\zeta^2}\left[\frac{1}{\zeta}e^{-\zeta|\tau|} - \frac{1}{\omega}e^{-\omega|\tau|}\right] \tag{48}$$

and the mean-square error is evaluated by letting τ equal zero,

$$\gamma_{\varepsilon\varepsilon}(\tau=0) = \widetilde{[{}^k\varepsilon(T_2)]^2} = \frac{\omega K}{\omega+\zeta} \tag{49}$$

Both ω and ζ are positive, therefore the mean-square error is less than K.

For the state variable switching system, ${}^kq(t)$ is compared to $Q(T_2)$, and the switch is positioned off when the state variable ${}^kq(t)$ equals the desired output $Q(T_2)$. In Figure 3 the error becomes,

$${}^k\varepsilon(T_2) = Q(T_2) - {}^ky(T_2) = [Q(T_2)-{}^kq(T_2)] - {}^kn(T_2) \tag{50}$$

and the mean-square error is,

$$\widetilde{[{}^k\varepsilon(T_2)]^2} = \widetilde{[n(T_2)]^2} \tag{51}$$

With the autocorrelation function of Equation (47), the mean-square error is,

$$\widetilde{[{}^k\varepsilon(T_2)]^2} = \gamma_{nn}(\tau=0) = K \tag{52}$$

For this case, the mean-square error is always less for the optimum control system than for the state variable switching system.

For unit white noise, the autocorrelation function is given by,

$$\gamma_{nn}(\alpha-\beta) = \delta(\alpha-\beta) \tag{53}$$

When the initial error is zero, substitution of Equation (53) into (46) yields,

$$\widetilde{[\varepsilon(T_2)]^2} = \gamma_{\varepsilon\varepsilon}(\tau=0) = \frac{\omega}{3}\tanh^3(\omega T_2) \tag{54}$$

Because the hyperbolic tangent never exceeds unity, the mean-square error has an upper bound equal to one third of the square root of the ratio of the output error weighting factor to the input error weighting factor. For comparison, the mean-square error in the state variable switching system with white noise is unbounded for all ω.

SECOND ORDER SYSTEM

The second order system may be given as,

$$\frac{d^2q}{dt^2} = m(t) \tag{55}$$

The optimum control law for this system is derived in APPENDIX D and is,

$$m^*(t) = M(t) + \psi_3(t)[Q(t)-q^*(t)] + \psi_4(t)\left[\frac{dQ}{dt} - \frac{dq^*}{dt}\right] \tag{56}$$

where:

$$\psi_3(t) = \omega\,\frac{\sin^2 z+\tanh^2 z\cos^2 z}{1 + \operatorname{sech}^2 z\cos^2 z},$$

$$\psi_4(t) = \sqrt{2\omega}\,\frac{\tanh z-\operatorname{sech}^2 z\,\cos z\,\sin z}{1 + \operatorname{sech}^2 z\,\cos^2 z},$$

$z = \sqrt{\frac{\omega}{2}}\,(T-t)$ and T is the terminal time.

The desired output and input relation is,

$$\frac{d^2Q}{dt^2} = M(t) \tag{57}$$

If the second derivative of the reference trajectory Q(t) exists everywhere except at a few isolated points, then the reference input M(t) is piece-wise continuous and can be determined. Thus the control law given by Equation (56) with an unspecified continuous M(t) is very general. The analysis may also be extended to the case where M(t) is piece-wise continuous.

DISCUSSION

The optimization problem in which the desired input is piece-wise continuous and the system is described by a linear constant coefficient system of differential equations has been shown to be solvable as a closed form control law.

The general result was applied to the simplified nuclear reactor kinetics equations with a specialized piece-wise continuous desired input. The results are identical with those obtained previously using the methods of calculus of variations and dynamic programming.

The closed form optimum control system is superior to the state variable switching system in the following particulars:

a) When random noise is added to the output, the optimum control system tends to have a smaller mean-square error than the state variable switching system.

(b) For nuclear rocket control, due to the inertia of the control rods, abrupt changes in position required by the state variable switching system, are physically impossible. The continuous closed form optimum control system leads to a more accurate physical representation of the control rods.

(c) The optimum control appears to have greatest potential in the field of nuclear propelled rockets. For this application, noise introduced into the output will affect the rocket thrust. Noise in the state variable switching system may cause switching at a time different from originally scheduled and may completely change the thrust program. The closed form optimum control system will tend to return to the original thrust program at the right time while the noise subsides.

As illustrated by the second order system, the optimum control law is quite general with regard to the reference trajectory. This is advantageous in that the derivation of the time varying gains required in the feedbacks is independent of the reference trajectory if the desired input is continuous.

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APPENDIX A

Pontryagin Maximum Principle(6),(7)

A system operates between time $\mu=0$ and $\mu=T_2-t$. The system is described by a set of differential equations,

$$\frac{dx_i(t;\mu)}{d\mu} = f_i(x_1,x_2,\ldots x_j\ldots x_n,x_{n+1}, u_1,u_2\ldots u_k\ldots u_r,\mu) \qquad \text{(A-1)}$$

$$x_i(t;0) = x_i^0(t) \qquad i = 1,2,\ldots n,n+1$$

where $x_i(t;\mu)$ are state variables, t is treated as a parameter and u_k are control variables. The problem is to minimize,

$$S(t) = \sum_{i=1}^{n+1} c_i x_i(t;\mu=T_2-t) \qquad \text{(A-2)}$$

with respect to the control variables. Constraints may exist on the control variables, $\bar{u}(t;\mu)\epsilon U$, where U is the admissible class of control functions in r space, and $\bar{u}(t;\mu)$ is the vector control function with components $u_k(t;\mu)$.

An auxiliary variable $p_i(t;\mu)$ is defined,

$$\frac{dp_i(t;\mu)}{d\mu} = -\sum_{j=1}^{n+1} p_j(t;\mu)\frac{\partial f_j}{\partial x_i} \qquad \text{(A-3)}$$

The free terminal condition or natural boundary conditions are,

$$p_i(t;\mu=T_2-t) = -c_i \qquad i = 1,2,\ldots n,n+1. \qquad \text{(A-4)}$$

A Hamiltonian is defined,

$$H[\bar{x}(t;\mu),\bar{u}(t;\mu),\mu] = \sum_{i=1}^{n+1} p_i(t;\mu)f_i[\bar{x}(t;\mu),\bar{u}(t;\mu),\mu] \qquad \text{(A-5)}$$

A sufficient condition for a minimum of S is that H be maximized with respect to the control vector at all times.

The sets of differential equations for the system and for the auxiliary variables are related to H by,

$$\frac{dx_i(t;\mu)}{d\mu} = \frac{\partial H}{\partial p_i} \quad \text{and} \quad \frac{dp_i(t;\mu)}{d\mu} = -\frac{\partial H}{\partial x_i} \tag{A-6}$$

APPENDIX B

Optimum Control Law for First Order Systems

The first piece of the piece-wise continuous input solution (prior to time T_1) is given by the solution to Equation (24),

$$p_1(t;\mu) = A_a \cosh(\omega\mu) + B_a \sinh(\omega\mu) \qquad t < T_1 \tag{B-1}$$

Substituting Equation (B-1) into (19) with $n = 1$ gives,

$$x_1(t;\mu) = Q_a(t+\mu) + \frac{\omega}{2\xi}[A_a \sinh(\omega\mu) + B_a \cosh(\omega\mu)] \tag{B-2}$$

With the equivalence between $x_1(t;\mu)$ and $q(t+\mu)$ expressed by Equation (6), (B-2) with (3) becomes,

$$q^*(\sigma) = Q_a(\sigma) + \frac{\omega}{2\xi}\left\{A_a \sinh[\omega(\sigma-t)] + B_a \cosh[\omega(\sigma-t)]\right\} \tag{B-3}$$

The constant B_a is evaluated by requiring $q(\sigma)$ to be equal to the measured value $q^*(t)$ at $\sigma = t$,

$$B_a = -\frac{2\xi}{\omega}[Q_a(t)-q^*(t)] \tag{B-4}$$

and,

$$q^*(\sigma) = Q_a(\sigma) + \frac{A_a}{2}\,\frac{\omega}{\xi} \sinh[\omega(\sigma-t)] - [Q_a(t)-q^*(t)] \cosh[\omega(\sigma-t)] \tag{B-5}$$

The last two boundary conditions are that the output signal and its derivative be continuous at time T_1. Equating Equations (32) and (B-5), and also equating their derivatives at $\sigma=T_1$ and eliminating B_b to solve for A_a gives,

$$\frac{A_a}{2\xi} = \frac{1}{\omega^2\psi_2}\left\{\frac{dQ_b(\sigma)}{d\sigma}\bigg|_{\sigma=T_1} - \frac{dQ_a(\sigma)}{d\sigma}\bigg|_{\sigma=T_1} - \omega[Q_a(T_1)-Q_b(T_1)]\theta + [Q_a(t)-q^*(t)]\psi_1\right\} \tag{B-6}$$

where:

$$\theta = \tanh[\omega(T_2-T_1)]$$

$$\psi_1 = \omega\left\{\sinh[\omega(T_1-t)] + \theta\cosh[\omega(T_1-t)]\right\}$$

$$\psi_2 = \cosh[\omega(T_1-t)] + \theta\sinh[\omega(T_1-t)]$$

Substituting Equation (B-4) into (B-1) gives,

$$p_1(t;\mu) = -\frac{2\xi}{\omega}[Q_a(t)-q^*(t)]\sinh(\omega\mu) + A_a\cosh(\omega\mu) \tag{B-7}$$

Substituting Equations (B-6) and (B-7) into (13) with n=1, using (15) and with the change in variables given by (3) and (6), at $\sigma=t$,

$$m^*(t)=M_a(t) + \frac{1}{\psi_2}\left\{\frac{dQ_b(\sigma)}{d\sigma}\bigg|_{\sigma=T_1} - \frac{dQ_a(\sigma)}{d\sigma}\bigg|_{\sigma=T_1} - \omega\theta[Q_a(T_1)-Q_b(T_1)] + [Q_a(t)-q^*(t)]\psi_1\right\} \quad t < T_1 \tag{B-8}$$

APPENDIX C

Nuclear Reactor Kinetics

The one delay group reactor kinetics equations for a nuclear reactor which is controlled by a black absorber are given by,[8]

$$\frac{d\phi}{dt} = \frac{\Delta K-\beta}{\Omega}\phi + \lambda c \tag{C-1}$$

$$\frac{dc}{dt} = \frac{\beta}{\Omega}\,\phi - \lambda c \tag{C-2}$$

where: ϕ = spatially independent neutron flux,

c = equivalent concentration of precursors for the one delay group case,

ΔK = reactivity,

β = fraction of total number of fission neutrons which are delayed,

Ω = neutron generation time, the average time before one neutron generates one prompt neutron or one precursor,

λ = equivalent decay constant for the one delay group case.

For reactivity less than one dollar $\frac{d\phi}{dt}$ may be neglected in Equation (C-1), see for example Smets[9]. With this approximation, elimination of c between Equations (C-1) and (C-2) gives,

$$\frac{dx}{dt} - \frac{1}{(1-\rho)\lambda}\frac{d\rho}{dt} = \frac{\rho}{1-\rho} \tag{C-3}$$

where $\rho = \Delta K/\beta$ and $\frac{dx}{dt} = \frac{1}{\lambda\phi}\frac{d\phi}{dt}$.

Defining q as a function of n and ρ,

$$q = \frac{1}{\lambda}\,\ln\left[\frac{(1-\rho)\,\phi}{(1-\rho_o)\,\phi o}\right] \tag{C-4}$$

where ρ_o and ϕ_o are reference values of ρ and ϕ. Differentiating Equation (C-4) and referring to the definition of $\frac{dx}{dt}$, Equation (C-3) may be written,

$$\frac{dq}{dt} = m \tag{C-5}$$

where $m = \frac{\rho}{1-\rho}$.

A common reactor start-up sequence is as follows: the power is constant at a low level prior to time $\sigma = 0$. At $\sigma = 0$ it is desired to increase the power at a constant period $(\phi/\frac{d\phi}{dt})$ until a desired power level is reached and thereafter maintain the power constant. The desired response for this start-up program is similar to Figure 1.

$$\begin{aligned} Q_a(\sigma) &= a\sigma & 0 \leq \sigma < T_1 \\ Q_b(\sigma) &= aT_1 & T_1 \leq \sigma \leq T_2 \end{aligned} \qquad \text{(C-6)}$$

where T_1 is a fixed intermediate time and "a" is a constant. Consistent with the fact that the desired input and output will satisfy the differential equation, the desired inputs are piece-wise continuous,

$$\begin{aligned} M_a(\sigma) &= a & 0 \leq \sigma < T_1 \\ M_b(\sigma) &= 0 & T_1 \leq \sigma \leq T_2 \end{aligned} \qquad \text{(C-7)}$$

The piece-wise desired input has a physical interpretation in terms of the control rod position. When the desired output is constant reactor period, the desired position of the control rods is that position which results in constant positive reactivity. When the desired output is constant power level, the desired position of the control rods is for zero reactivity.

APPENDIX D

Optimum Control Law for Second Order Systems

The specialized second order system (n=2) which represents the attitude control problem[10] is Equation (2) with $a_o = a_1 = 0$ and $a_2 = 1$. Equation (25) becomes,

$$\frac{d^4 p_2(t;\mu)}{d\mu^4} + \omega^2\, p_2(t;\mu) = 0 \qquad \text{(D-1)}$$

The solution is,

$$p_2(t;\mu) = A \cosh\beta_1 \cos\beta_1 + B \cosh\beta_1 \sin\beta_1 + C \sinh\beta_1 \cos\beta_1 + D \sinh\beta_1 \sin\beta_1 \tag{D-2}$$

where A,B,C and D are constants, and $\beta_1 = \sqrt{\frac{\omega}{2}}\,\mu$.

From Equations (A-4) and (9) two boundary conditions are,

$$p_1(t;\mu=T-t) = p_2(t;\mu=T-t) = 0 \tag{D-3}$$

where T is the time at which control is to terminate. The successive auxiliary variables are related by Equation (14),

$$p_1(t;\mu) = \frac{dp_2(t;\mu)}{d\mu} \tag{D-4}$$

The two boundary conditions give the relations,

$$A + B \tan z + C \tanh z + D \tanh z \tan z = 0 \tag{D-5}$$

and

$$(A+D)\tanh z + (D-A)\tan z + (B-C)\tanh z \tan z + B+C = 0 \tag{D-6}$$

where $z = \sqrt{\frac{\omega}{2}}\,(T-t)$.

From Equation (19) with n=2, and the definition of $q(t+\mu) = x_1(t;\mu)$ there results,

$$q(t+\mu) = Q(t+\mu) - \frac{1}{2\xi}\,\frac{d^2p_2(t;\mu)}{d\mu^2} \tag{D-7}$$

Substituting Equation (D-2) into (D-7) gives,

$$q(t+\mu) = Q(t+\mu) + \frac{\omega}{2\xi}\begin{bmatrix} A \sinh\beta_1 \sin\beta_1 \\ - B \sinh\beta_1 \cos\beta_1 \\ + C \cosh\beta_1 \sin\beta_1 \\ - D \cosh\beta_1 \cos\beta_1 \end{bmatrix} \tag{D-8}$$

The remaining two boundary conditions are that the output and its derivative are known at the present time t,

$$q(\sigma)\Big|_{\sigma=t} = q^*(t)$$

$$\left.\frac{dq(\sigma)}{d\sigma}\right|_{\sigma=t} = \frac{dq^*(t)}{dt} \tag{D-9}$$

The optimum control law is obtained from Equation (13) (with n=2) at the present time $\sigma=t$, which is equivalent to $\mu = \beta_1 = 0$. Substituting Equation (D-2) into (13) and the definition of $m(t+\mu) = u(t;\mu)$ gives,

$$m^*(t) = M(t) + \frac{A}{2\nu} \tag{D-10}$$

Equation (D-8) with (3) is used to evaluate (D-9) giving,

$$D = 2\,\frac{\xi}{\omega}\,[Q(t)-q^*(t)] \tag{D-11}$$

$$B-C = \xi\left(\frac{2}{\omega}\right)^{3/2}\left[\frac{dQ}{dt} - \frac{dq^*}{dt}\right] \tag{D-12}$$

By adding and subtracting C to Equation (D-6), C can be solved in terms of A,D, and (B-C). Adding and subtracting C tan z to Equation (D-5) gives a relationship between A,B-C,C and D. By substituting the expression for C from the modified Equation (D-6) and using trigonometric identities, the following result is obtained,

$$A = D\,\frac{\tanh^2 z+\tan^2 z}{\operatorname{sech}^2 z+\sec^2 z} + (B-C)\,\frac{\tanh z\sec^2 z-\operatorname{sech}^2 z\tan z}{\operatorname{sech}^2 z+\sec^2 z} \tag{D-13}$$

The optimum control law, obtained by substituting Equations (D-11), (D-12) and (D-13) into (D-10) is given as Equation (56.).

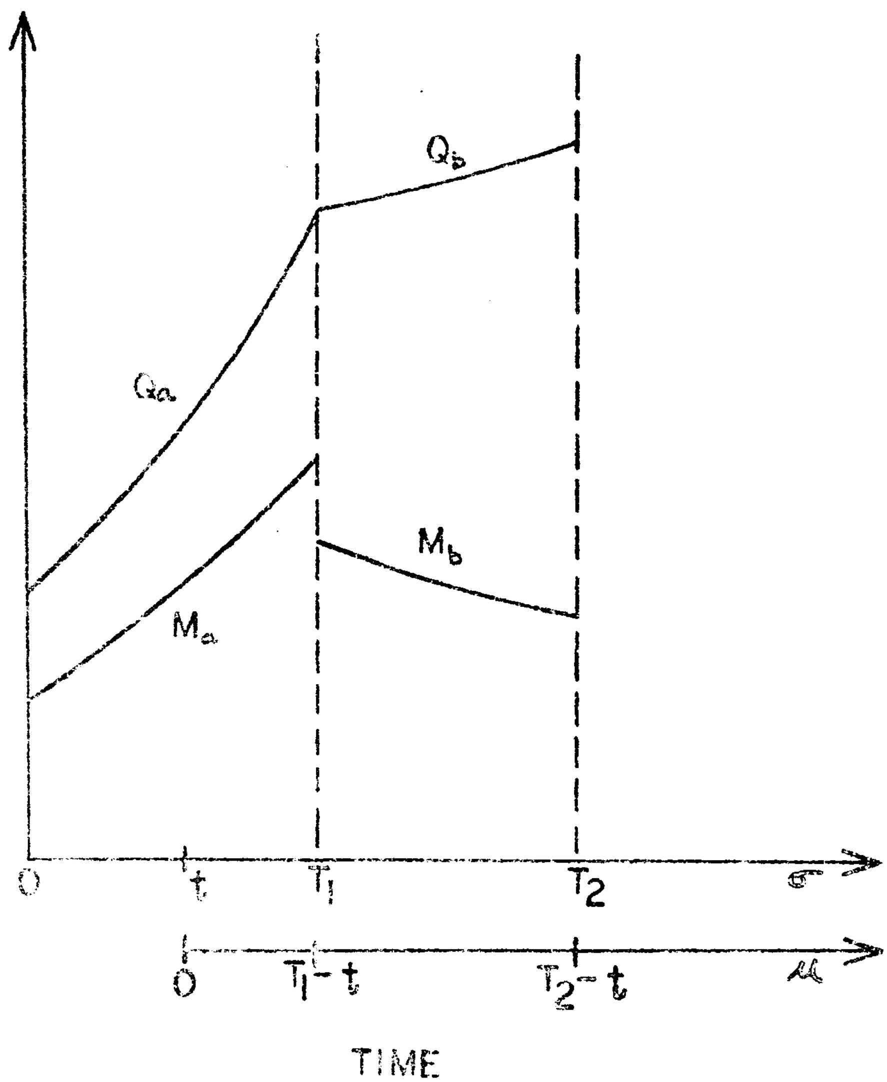

FIGURE 1

ILLUSTRATION OF DESIRED INPUT AND OUTPUT, AND TIME SCALES

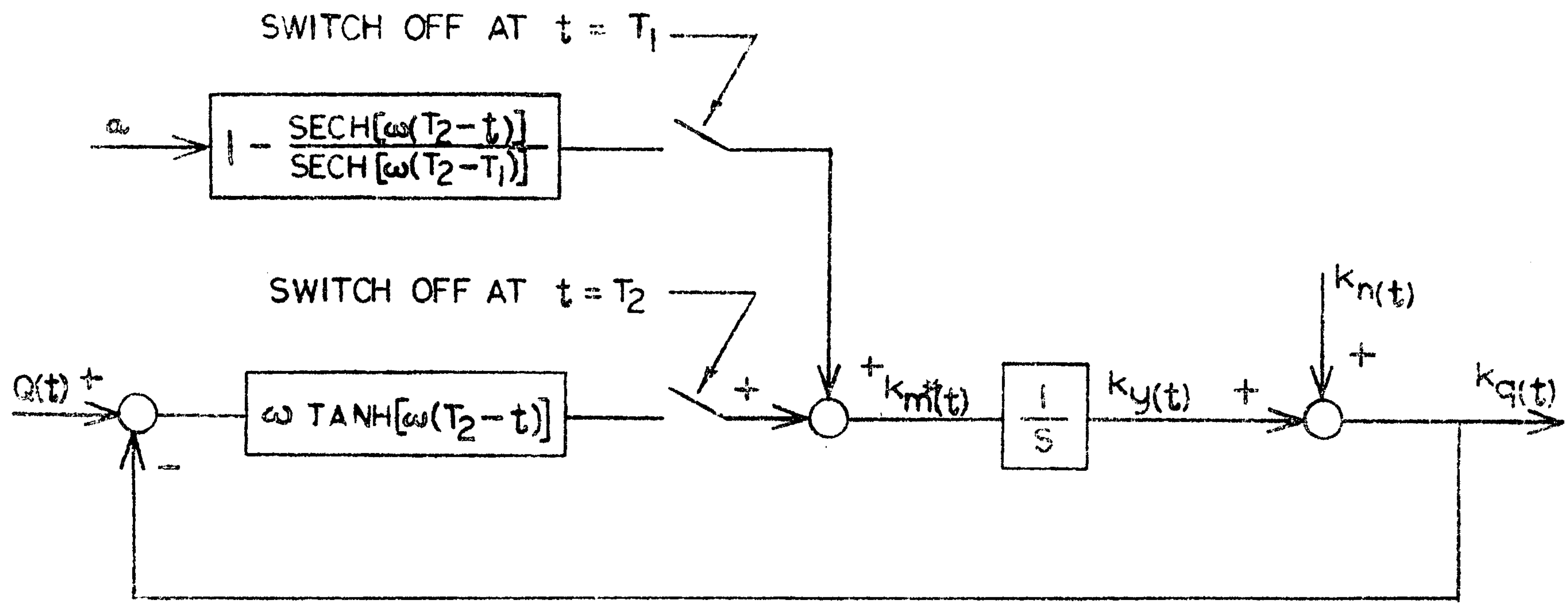

FIGURE 2

OPTIMUM CONTROL SYSTEM WITH
ADDITIVE NOISE IN THE OUTPUT

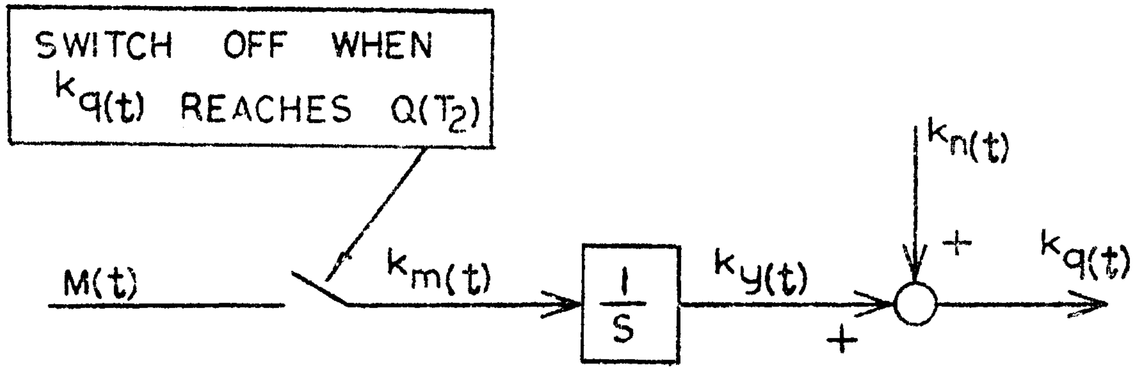

FIGURE 3.

OPTIMUM STATE VARIABLE SWITCHING SYSTEM
WITH ADDITIVE NOISE IN THE OUTPUT